# Effect of vorticity on pre-breaking waves in shallow water

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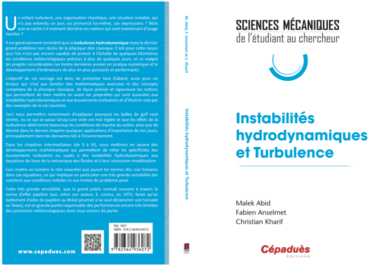
# Collaborations with Fabien

I would like to mention three scientific collaborations with Fabien:

- Co-supervisor of the PhD thesis of S. Ravier: S. Ravier, M. Abid, M. Amielh, F. Anselmet, Direct numerical simulations of variable density plane jets, Journal of Fluid Mechanics, 546, 153-191 (2006)
- Co-author of the book: Instabilités hydrodynamiques et Turbulence, M. Abid, F. Anselmet, C. Kharif, Cépaduès ed. (2017)




- Co-supervisor of the PhD thesis of R. Vallon: Liquid jet atomization small and large turbulent scales (2018–)

# Overview

1. Motivations
2. Model equations with vorticity
3. Numerical method and validation
   - Validation of the numerical method
   - Validation of the model (Experiments, Favre-waves)
4. The effect of vorticity on the pre-breaking of gravity waves
   - Whitham equation with vorticity (Vor-Whitham)
   - Generalized Whitham equation (Gen-Whitham)
5. Conclusions

# Motivations: breaking of gravity waves

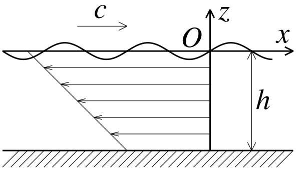
- Ultimately gravity waves breaks: turbulence is generated and plays a significant role in the interaction of water waves with the atmosphere.
- Wave-breaking still an open problem of nonlinearity.

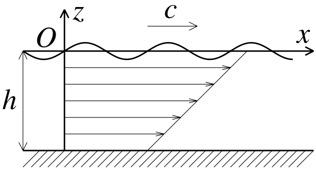
Plunging breaker (www.wikipedia.fr)


Spilling breaker (www.geology.uprm.edu)

- Characterization: a) kinematically: $u > c$, b) geometrically: infinite wave-profile slope, i.e. a singularity. Only pre-breaking here.

# Motivations




Generally, in coastal and ocean waters, current velocity profiles are established by bottom friction and wind stress at the sea surface, and consequently are vertically varying $\rightarrow$ Vorticity.

- Purpose: to derive an approximate model to investigate nonlinear long wave dynamics in the presence of a vertically sheared current of constant Vorticity.
- Constant vorticity: the first approximation (simplification) that should be tried.

# Motivations: breaking of gravity waves

- The KdV equation is not the appropriate model for describing wave evolution to breaking.
- Whitham (1974) suggested a model equation (without vorticity) for breaking.
- The Whitham equation and KdV equation have the same (weak) nonlinear term and different dispersive terms.
- The dispersive term of the Whitham equation corresponds to exact linear dispersion and consequently allows the introduction of small scales which are important in the breaking phenomenon.
- We will generalize the Whitham equation to the fully nonlinear case, for flows with constant vorticity (Gen-Whitham equation), and obtain the Whitham equation with constant vorticity in the weakly nonlinear limit (Vor-Whitham equation).

# Model equations with vorticity

- Freeman & Johnson (1970) $\Longrightarrow$ KdV eq.
- Choi (2003) $\Longrightarrow$ GN eqs. $\Longrightarrow$ Boussinesq eqs. $\Longrightarrow$ KdV eq.
- Johnson (2012) $\Longrightarrow$ Boussinesq eqs. and Camassa-Holm eq.
- Richard & Gavriluyk (2015) $\Longrightarrow$ Generalised GN eqs. (shear flows and turbulence).
- Castro & Lannes (2014) $\Longrightarrow$ GN eqs. with general vorticity.
- Kharif and Abid (2018) $\Longrightarrow$ Generalized Whitham equation with constant vorticity: one equation model for the free surface: *Nonlinear water waves in shallow water in the presence of constant vorticity: A Whitham approach*, Eur. J. Mechanics / B Fluids 72 (2018) 12-22.
    - Fully nonlinear
    - Fully linearly dispersive

# Model equations with vorticity

- St Venant equations (shallow water equations) in the presence of a vertically sheared current: $U = U_0 + \Omega z$

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}[u(\eta + h) + \frac{\Omega}{2}\eta^2 + U_0\eta] = 0 \qquad (1)$$

$$\frac{\partial u}{\partial t} + (u + U_0 - \Omega h)u_x + g\eta_x = 0 \qquad (2)$$

$U_0$: velocity at the free surface $z = 0$ of the underlying current.
$\eta(x, t)$: free surface elevation.
$u(x, t)$: longitudinal component of the wave induced velocity.
$\Omega$: shear intensity (the vorticity is $-\Omega$)
$h$: water depth at rest.
$g$: gravity.

# Model equations with vorticity

- The hyperbolic equations (1) and (2) admit the following Riemann invariants:

$$u + \frac{\Omega H}{2} \pm \left\{ \sqrt{gH + \Omega^2 H^2/4} + \frac{g}{\Omega} \ln \left[ 1 + \frac{\Omega}{2g} (\Omega H + 2\sqrt{gH + \Omega^2 H^2/4}) \right] \right\} = \mathrm{constant}$$

on characteristic lines

$$\frac{dx}{dt} = u + U_0 + \frac{1}{2}\Omega(\eta - h) \pm \sqrt{gH + \frac{\Omega^2 H^2}{4}} \qquad (3)$$

where $H = \eta + h$.

- For $\Omega \to 0$ and $U_0 = 0 \implies u \pm 2\sqrt{gH} = \mathrm{cst}$ on $dx/dt = u \pm \sqrt{gH}$

# Model equations with vorticity

- The constant is determined for $u = 0$ and $\eta = 0$ $(H = h)$.
- Finally, Riemann invariants with vorticity read:

$$\begin{aligned} u + \frac{\Omega\eta}{2} \quad &+ \quad \sqrt{gH + \Omega^2 H^2/4} - \sqrt{gh + \Omega^2 h^2/4} \\ &+ \quad \frac{g}{\Omega} \ln \left[ \frac{1 + \frac{\Omega}{2g}(\Omega H + 2\sqrt{gH + \Omega^2 H^2/4})}{1 + \frac{\Omega}{2g}(\Omega h + 2\sqrt{gh + \Omega^2 h^2/4})} \right] = 0 \end{aligned} \quad (4)$$

$$\begin{aligned} u + \frac{\Omega\eta}{2} \quad &- \quad \sqrt{gH + \Omega^2 H^2/4} + \sqrt{gh + \Omega^2 h^2/4} \\ &- \quad \frac{g}{\Omega} \ln \left[ \frac{1 + \frac{\Omega}{2g}(\Omega H + 2\sqrt{gH + \Omega^2 H^2/4})}{1 + \frac{\Omega}{2g}(\Omega h + 2\sqrt{gh + \Omega^2 h^2/4})} \right] = 0 \end{aligned} \quad (5)$$

# Model equations with vorticity

- Let us consider a wave moving rightwards:

$$u = -\frac{\Omega\eta}{2} + \sqrt{gH + \Omega^2 H^2/4} - \sqrt{gh + \Omega^2 h^2/4} + \frac{g}{\Omega} \ln\left[\frac{1 + \frac{\Omega}{2g}(\Omega H + 2\sqrt{gH + \Omega^2 H^2/4})}{1 + \frac{\Omega}{2g}(\Omega h + 2\sqrt{gh + \Omega^2 h^2/4})}\right] \qquad (6)$$

- For $\Omega \to 0$ we have $u \to 2\sqrt{gH} - 2\sqrt{gh}$

# Model equations with vorticity: Vor-Riemann equation

- Substituting the expression $u(x,t)$ into the conservation of mass equation (1) gives (Vor-Riemann equation):

$$\eta_t + \Big\{ U_0 - \frac{\Omega h}{2} + 2\sqrt{g(\eta+h)+\Omega^2(\eta+h)^2/4} - \sqrt{gh+\Omega^2 h^2/4} +$$

$$\frac{g}{\Omega}\ln\left[1+\frac{\Omega}{2g}\frac{\Omega\eta+2(\sqrt{g(\eta+h)+\Omega^2(\eta+h)^2/4}-\sqrt{gh+\Omega^2h^2/4})}{1+\frac{\Omega}{g}(\frac{\Omega h}{2}+\sqrt{gh+\Omega^2h^2/4})}\right]\Big\}\eta_x = 0 \tag{7}$$

- For $U_0 = 0$ and $\Omega \to 0$ equation (7) reduces to:

$$H_t + (3\sqrt{gH} - 2\sqrt{gh})H_x = 0, \quad \text{with} \quad H = \eta + h. \tag{8}$$

# Model equations with vorticity: Gen-Whitham equation

- Following Whitham (1974) and Drazin & Johnson (1989) the full linear dispersion is introduced heuristically:

$$\eta_t + \Bigg\{ U_0 - \frac{\Omega h}{2} + 2\sqrt{g(\eta+h) + \Omega^2(\eta+h)^2/4} - \sqrt{gh + \Omega^2 h^2/4} +$$

$$\frac{g}{\Omega} \ln\left[1 + \frac{\Omega}{2g} \frac{\Omega\eta + 2(\sqrt{g(\eta+h) + \Omega^2(\eta+h)^2/4} - \sqrt{gh + \Omega^2 h^2/4})}{1 + \frac{\Omega}{g}(\frac{\Omega h}{2} + \sqrt{gh + \Omega^2 h^2/4})}\right] \Bigg\} \eta_x +$$

$$K * \eta_x = 0. \qquad (9)$$

- where $K * \eta_x$ is a convolution product. The kernel K is given as the inverse Fourier transform of the linear phase velocity of gravity waves in finite depth in the presence of constant vorticity $\Omega$: $K = F^{-1}(c)$ with

$$c = U_0 - \frac{\Omega \tanh(kh)}{2k} + \sqrt{\frac{g \tanh(kh)}{k} + \frac{\Omega^2 \tanh^2(kh)}{4k^2}}.$$

# KdV equation with vorticity

- For weakly nonlinear ($\eta/h \ll 1$) and weakly dispersive ($kh \ll 1$) equation (9) reduces to the KdV equation with vorticity (with $U_0{=}0$)

$$\eta_t + c_0(\Omega)\eta_x + c_1(\Omega)\eta\eta_x + c_2(\Omega)\eta_{xxx} = 0, \tag{10}$$

  with

$$\begin{aligned} c_0 &= -\frac{\Omega}{2} + \sqrt{1 + \Omega^2/4}, \\ c_1 &= \frac{3 + \Omega^2}{\sqrt{4 + \Omega^2}}, \\ c_2 &= \frac{2 + \Omega^2 - \Omega\sqrt{4 + \Omega^2}}{6\sqrt{4 + \Omega^2}}. \end{aligned}$$

  The KdV equation (10) has been normalized so that $h = 1$ and $g = 1$ and is identical to those derived by Freeman & Johnson (1970) and Choi (2003) who used multiple scale methods.

# Whitham equation with vorticity: Vor-Whitham

- For weakly nonlinear waves ($\eta/h \ll 1$) the Gen-Whitam equation (9) becomes the Whitham equation with constant vorticity (Vor-Whitham) given by

$$\eta_t + \frac{3gh + h^2\Omega^2}{h\sqrt{4gh + h^2\Omega^2}}\eta\eta_x + K * \eta_x = 0 \tag{11}$$

- Note that in the Whitham equation the exact linear dispersion is considered unlike the KdV equation

# Validation of the numerical method

Numerical integration of equations Gen-Whitham (9), Vor-Kdv (10) and (11)

- Pseudo-spectral method (number of grid points $N_x = 2^{14}$)
- Runge-Kutta of order 4 ($\Delta t = 0.005, g = 1, h = 1$)
- KdV Invariants (mass, momentum and energy) are conserved with a relative accuracy of $O(10^{-9})$

# Validation of the numerical method: Dam-Break

- For $U_0 = 0$ and $\Omega \to 0$ the Vor-Riemann equation (7) reduces to

$$H_t + (3\sqrt{gH} - 2\sqrt{gh})H_x = 0, \quad \text{with} \quad H = \eta + h. \qquad (12)$$

For $t > 0$, the nonlinear analytical solution of equation (12) is

$$\begin{aligned}
H(x,t) &= h, & u(x,t) &= 0; & \frac{x}{t} &\geq \sqrt{gh} \\
H(x,t) &= \frac{h}{9}\left(2 + \frac{x}{\sqrt{gh}\ t}\right)^2, & u(x,t) &= -\tfrac{2}{3}\left(\sqrt{gh} - \tfrac{x}{t}\right); & -2\sqrt{gh} \leq \frac{x}{t} &\leq \sqrt{gh} \\
H(x,t) &= 0, & u(x,t) &= 0; & \frac{x}{t} &\leq -2\sqrt{gh} \qquad (13)
\end{aligned}$$

- At time $t = 0$ the initial condition is $H(x,0) = h(1 + \tanh(2x))/2$ and $u(x,0) = 0$ everywhere

# Numerical validation: Dam-Break

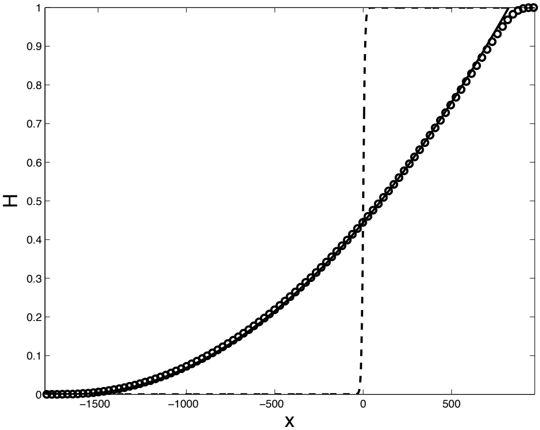



Figure: Dam-break: comparison between analytical (solid line) and numerical solutions ($\circ$) after the dam has broken. The dashed line represents the initial condition at $t = 0$ ($h = 1,\ g = 1$).

# Validation of the Gen-Whitham model: Favre-waves

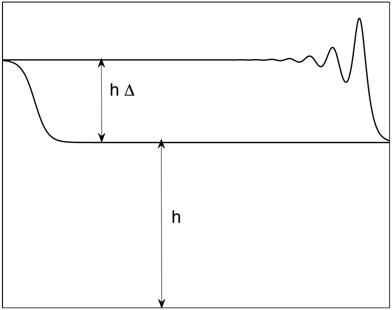



Figure: Sketch of the evolution of an undular bore from its initial position

- An undular bore is formed when a sudden discharge of water at rest of depth $h(1+\Delta)$ is initiated into still water of depth $h$. The bore is the region of transition between two uniform depths.

# Validation of the Gen-Whitham model: Favre-waves

- Favre's experiments (1935) at Ecole Polytechnique Fédérale de Zurich
  - Canal dimensions: 73.58 m long, 0.42 m wide and 0.40 m height.
  - Water depth: h= 20 cm or h=10 cm
  - For $\Delta < 0.28 \implies$ non breaking undular bores
  - For $\Delta > 0.28 \implies$ breaking undular bores

# Validation of the Gen-Whitham model: Favre-wave

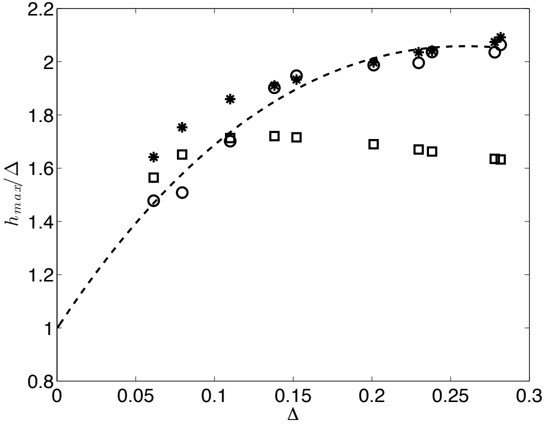



Figure: Dimensionless height of the leading wave as a function of the initial relative difference in water level. Favre's experiments ($\circ$), Gen-Whitham equation (9) with damping ($*$), KdV equation with damping ($\square$). $(U_0, \Omega) = (0, 0)$.

# Validation of the Gen-Whitham model: Favre-wave

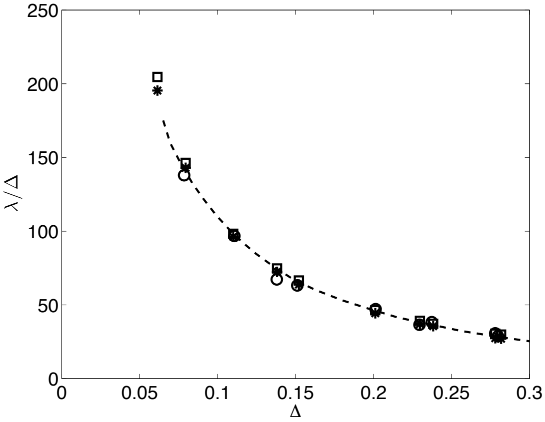



Figure: Dimensionless wavelength of the trailing waves as a function of the initial relative difference in water level. Favre's experiments ($\circ$), Gen-Whitham equation (9) with damping ($*$), KdV equation with damping ($\Box$). $(U_0, \Omega) = (0, 0)$.

# Breaking time for hyperbolic waves and dispersive waves in the presence of vorticity

- The Vor-Riemann equation (7) (hyperbolic equation) in dimensionless form ($g = 1,\ h = 1$) can be rewritten as follows

$$\eta_t + \mathcal{C}(\eta)\eta_x = 0 \qquad (14)$$

$$\mathcal{C}(\eta) = -\frac{\Omega}{2} + 2\sqrt{(\eta+1) + \Omega^2(\eta+1)^2/4} -$$

$$\sqrt{1+\Omega^2/4} + \frac{1}{\Omega}\ln\left[1 + \frac{\Omega}{2}\,\frac{\Omega\eta + 2(\sqrt{(\eta+1)+\Omega^2(\eta+1)^2/4} - \sqrt{1+\Omega^2/4})}{1+\Omega(\frac{\Omega}{2} + \sqrt{1+\Omega^2/4})}\right]$$

- Equation (14) is equivalent to the following system

$$\frac{d\eta}{dt} = 0, \quad \text{along the characteristic curve} \quad \frac{dx}{dt} = \mathcal{C}(\eta)$$

# Breaking time for hyperbolic waves with vorticity

- The characteristic curves are straight lines in the $(x, t)$-plane

$$x = x_0 + \mathcal{C}(\eta_0(x_0))t$$

$$x = x_0 + \mathcal{V}(x_0)t$$

- $\eta_0(x) = \eta(x, 0)$ is the initial condition.
- $x_0$ is the point where the characteristic curve intersects the $x$-axis $(t = 0)$
- the slope of the profile at $t$ is

$$\frac{\partial \eta}{\partial x} = \frac{d\eta_0/dx_0}{1 + \frac{d\mathcal{V}}{dx_0}t}$$

- On any characteristic for which $\frac{d\mathcal{V}}{dx_0} < 0$ the slope of the profile becomes infinite when $t = -(d\mathcal{V}/dx_0)^{-1}$.

# Breaking time for hyperbolic waves with vorticity

- Consequently, breaking wave first occurs on the characteristic curve intersecting the $x$-axis at $x_0 = x_{0_B}$ for which $\frac{d\mathcal{V}}{dx_0}(x_{0_B}) < 0$ with $|\frac{d\mathcal{V}}{dx_0}(x_{0_B})|$ is a maximum.
- The breaking time is

$$t_B = -(\frac{d\mathcal{V}}{dx_0}(x_{0_B}))^{-1} \tag{15}$$

- Herein, the breaking wave phenomenon can be understood as the blow-up of the slope in finite time $t_B$.

# Breaking time for dispersive waves

- For dispersive waves in shallow water, there is no analytical expression of the breaking time.
- The determination of the breaking time can be carried out numerically (Sulem et al., 1983).
- When $\eta(x,t)$ is an analytic function, its Fourier coefficients (with respect to $x$) decay faster than any power of $1/k$ ($k$ is the wavenumber) in the limit $k \to \infty$.
- When $\eta$ is singular, its Fourier coefficients decay algebraically with $1/k$.
- Hence, to detect the time of the appearance of the singularity we assume that the Fourier coefficients of the solution $\eta(x,t)$ behave as:

$$\hat{\eta}_k(t) = C(t)k^{-\alpha(t)}e^{-\delta(t)k}.$$

- The breaking time is defined as the time of vanishing of the analyticity strip: $\delta(t_B) = 0$.

# Validation: analyticity strip method

- This method is validated against the time of breaking when all the equations studied here are hyperbolic (vanishing dispersion).
- In those cases, the expression of the time of breaking could be obtained analytically as a function of $\Omega$:

$$t_B = -\left(\frac{d\mathcal{V}}{dx_0}(x_{0_B})\right)^{-1}$$

- The initial condition is

$$\eta(x,0) = a\cos(kx) + \frac{3-\sigma^2}{4\sigma^3}a^2k\cos(2kx+\varphi) \qquad (16)$$

where $\sigma = \tanh(kh)$, $h = 1$, $k = 1$, $a = 0.10$ and $\varphi = 0$.

# Validation: analyticity strip method

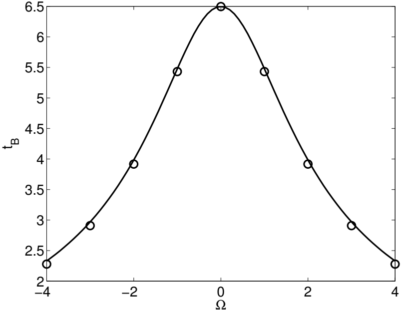



Figure: Theoretical and numerical breaking times as a function of the vorticity within the framework of Vor-Riemann equation (7). The solid line corresponds to the theoretical solution whereas the circles correspond to numerical values ($\delta(t_B) = 0$).

- The breaking time decreases when the shear intensity increases.

# Breaking time: Vor-Whitham

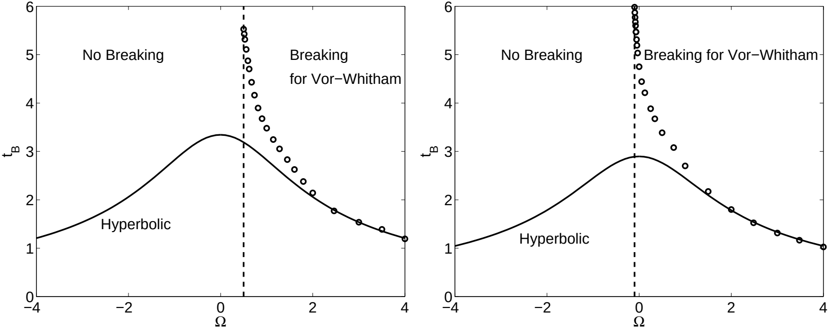



Figure: Breaking time as a function of the vorticity for the symmetric initial condition ($a = 0.16$, $\varphi = 0$) (left) and for the asymmetric initial condition ($a = 0.16$, $\varphi = 3\pi/2$) (right). The solid line corresponds to the hyperbolic Vor-Riemann equation (7) and the circle to the Vor-Whitham equation (11).

- For large values of Ω the model is hyperbolic in nature.
- There is a critical value of Ω for breaking to occur. Obviously, it depends on dispersion.
- Negative values of the vorticity stimulate the breaking phenomenon (opposing current).

# Breaking time: Gen-Whitham

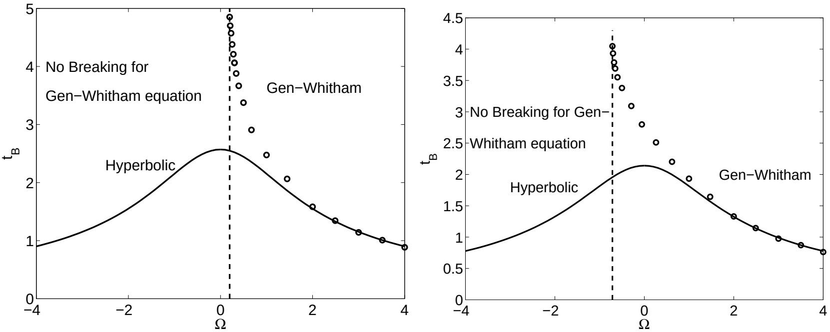



Figure: Breaking time as a function of the vorticity within the framework of the generalised Whitham equation with a symmetric initial condition ($a = 0.20$, $\varphi = 0$) (left) and an asymmetric initial condition ($a = 0.20$, $\varphi = 3\pi/2$) (right). The solid line corresponds to the hyperbolic case.

- Same conclusions as those for Vor-Whitham equation.
- The breaking time is smaller than that for Vor-Whitham equation.

# Pre-breaking profiles of the surface elevation

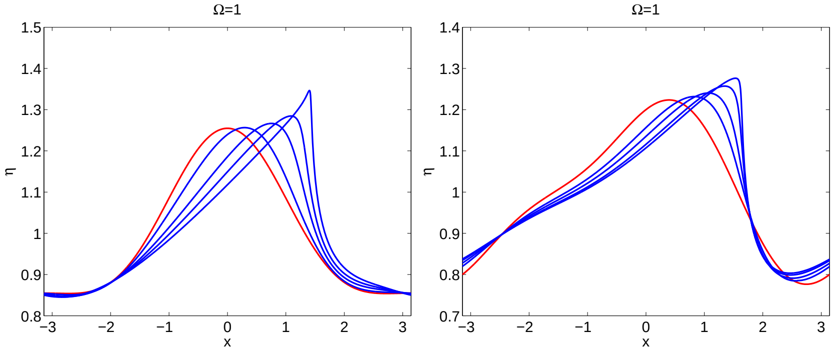



Figure: Time evolution of the initial symmetric profile ($a = 0.20$, $\varphi = 0$) (left) and asymmetric initial profile ($a = 0.20$, $\varphi = 3\pi/2$) (right) to breaking for $\Omega = 1$.

- Gen-Whitham equation, $g = 1$, $h = 1$.

# Conclusions

- We have derived a one equation model to investigate nonlinear long wave dynamics in the presence of a vertically sheared current of constant Vorticity.
- The model is validated.
- We have studied the effect of constant vorticity on the breaking of long gravity waves:
  - The sign of the vorticity is important: negative vorticity (opposing current) stimulate the breaking phenomenon.
  - For dispersive waves, there is a threshold of the vorticity value for the breaking to occur.
- Future work: we will use BIEM with constant vorticity to study the breaking of the undular bore using a geometric criterion, and will compare it with the kinematic criterion of breaking ($u = c$, collaboration with Prof. H. Kalisch, Univ. of Bergen, Norway).

The End

Thank you for attention

# Solitary and Cnoidal waves: vorticity effect

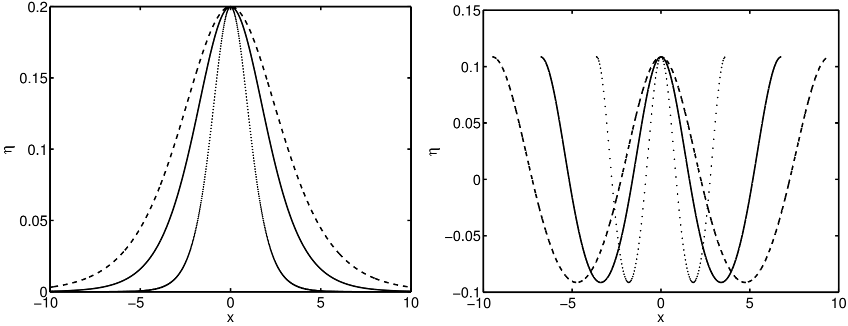



Figure: Profiles of solitary waves (left) and cnoidal waves (right) for various values of the vorticity. Solid line ($\Omega = 0$), dashed line ($\Omega = -1$) and dotted line ($\Omega = 1$).